APPENDIX

Proof for n-gram Outermost Angle Measure Sums

Let $x_1, x_2, x_3, \ldots, x_n$ be the degree measures of the outermost angles of any n-gram and let $a_1, a_2, a_3, \ldots, a_n$ be the degree measures of the interior n-gon (see **fig. 5**). Using the relationships used in the second approach for the pentagon, we obtain the following:

$$x_1° = a_n° + a_1° - 180°$$
$$x_2° = a_1° + a_2° - 180°$$
$$x_3° = a_2° + a_3° - 180°$$

and, in general,

$$x_n° = a_{n-1}° + a_n° - 180°.$$

Summing the equations yields

$$x_1° + x_2° + x_3° + \cdots + x_n° = 2(a_1 + a_2 + a_3 + \cdots + a_n)° - 180°\,n.$$

And, since

$$a_1° + a_2° + a_3° + \cdots + a_n° = 180°\,(n-2),$$

we can substitute to get

$$x_1° + x_2° + x_3° + \cdots + x_n° = 2(180°\,(n-2)) - 180°\,n,$$

which simplifies to

$$x_1° + x_2° + x_3° + \cdots + x_n° = 180°\,(n-4).$$

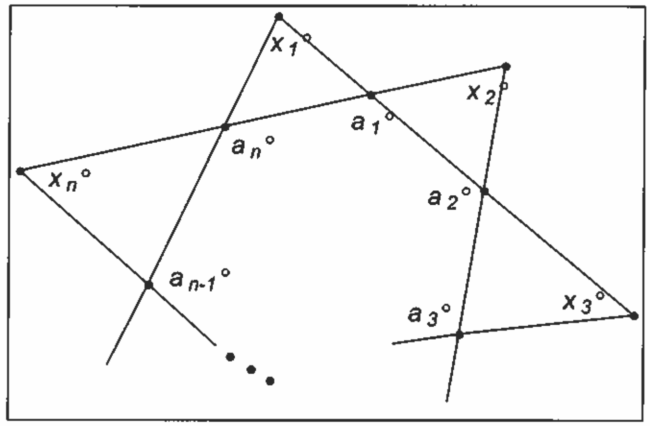

Fig. 5 The n-gram presents another level of generalization.

Al Cuoco, E. Paul Goldenberg, and June Mark

Organizing a Curriculum around Mathematical Habits of Mind

"Although it is necessary to infuse courses and curricula with modern content, what is even more important is to give students the tools they will need in order to use, understand, and even make mathematics that does not yet exist. A curriculum organized around habits of mind tries to close the gap between what the users and makers of mathematics do and what they say." (Cuoco, Goldenberg, and Mark 1996, p. 376)

The three articles in this cross-journal "Contemporary Curriculum Issues" series are written by the Educational Development Center curriculum development team of Cuoco, Goldenberg, Mark, and Sword. The team pioneered work using mathematical habits of mind that are central to the work of mathematicians for organizing school mathematics curricula. The quote above is from their classic article, "Habits of Mind: An Organizing Principle for Mathematics Curricula" (Cuoco, Goldenberg, and Mark 1996).

Regardless of the level at which you teach, you will find that each article's important message is relevant across the grades. The *TCM* article considers algebra's ideas, logic, techniques, and habits of mind as well as when and to what extent they can be learned with intellectual integrity in the elementary school grades before a formal course on algebra. In the *MTMS* article, the authors argue that developing mathematical habits of mind in the middle grades is essential for making the critical transition from arithmetic to algebra. The authors of the *MT* article reflect on their work in using the habits-of-mind approach for organizing high school curricula. They indicate that such an approach offers a vehicle for paring down the collection of methods and techniques one needs in high school, leaving a small set of general-purpose tools that tie together many seemingly different mathematical terrains.

Building coherence in the development of mathematical ideas across the grades is key to improving students' mathematical learning in the United States. Knowing the mathematical experiences, understanding, skills, and habits of mind that students bring to a grade level and what the expectations are for the following grades can help teachers bridge the transitions for students on each end.

This department provides a forum to stimulate discussion on contemporary curricular issues across a K-12 audience. NCTM has published sets of three articles, focused on a single curriculum issue. Each article addressed the issue from the perspective of the audience of the journal in which it appeared. Collectively, the articles were intended to increase communication and dialogue on issues of common interest related to curriculum.

In the years immediately following the release of *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989), there was a strong imperative to invigorate school mathematics. With government and private funding, teams around the country took a clean-slate approach to precollege mathematics and developed new curricula aimed at making more mathematics more meaningful and more accessible to more students. A central focus of these efforts was a shift toward student-centered, problem-based classrooms in which teachers support students as they work out the mathematics for themselves.

At the high school level, along with this shift in pedagogy came topical reorganizations—the elimination of topics considered less useful than in previous generations and the infusion of more "modern" topics that had found or that had shown potential to find applications in a variety of mathematics-related fields.

Around 1992, the authors started thinking about alternatives to this approach to reform. It seemed to us that the real utility of mathematics for many students, especially for those who would not go into STEM fields, came from a style of work—a web of ways of thinking about the world—that mathematicians use in their profession. Mathematicians have long realized that their methods are often just as important as their results. For example, William Thurston (1994) notes: "What mathematicians most wanted and needed from me was to learn my ways of thinking, and not in fact to learn my proof of the geometrization conjecture for Haken manifolds."

Our teaching experience and our work with teachers convinced us that raising the methods used by mathematicians to the same level of importance as the results of those methods would be a viable organizer for a high school curriculum. We made a detailed analysis of what we came to call *mathematical habits of mind*, and we made a case for using the ways of thinking that are indigenous to mathematics as a central benchmark both for deciding what topics to include in a high school curriculum and for determining how to develop them (Cuoco, Goldenberg, and Mark 1996).

Here we discuss the implications of the habits-of-mind approach for high school curricula, using the Center for Mathematics Education (CME) Project (EDC 2009) as a source of examples. In the course of developing this program, we found that using mathematical habits of mind as an organizer can bring genuine and often surprising coherence to a curriculum (Cuoco, Goldenberg, and Mark 2009). Although our original motivation was to help students make sense of the mathematical topics they study, the habits-of-mind approach also provided us with a vehicle for paring down the collection of methods and techniques one needs in high school, leaving a small set of general-purpose tools (Cuoco 2008) that tie together many seemingly different mathematical terrains.

HABITS OF MIND

We will examine general mathematical habits that are used across high school as well as specific geometric and algebraic habits. Good descriptions of

> **One goal of our curriculum materials is to make explicit the kinds of thought experiments that bring life to such classic contexts.**

statistical thinking and how it differs from mathematical thinking can be found in *Focus in High School Mathematics: Reasoning and Sense Making* NCTM (2009a) and in Cobb and Moore (1997).

General Mathematical Habits

These ways of thinking run throughout mathematics and can be developed within the contexts of algebra, geometry, probability, and statistics. They include the following.

Performing Thought Experiments

One can ask many questions about the classic context in which one starts with a rectangle of a certain size and cuts small congruent squares out of each corner, folding up the sides to make an open box. To reason about this context, one needs to be able to have a mental image, seeing, perhaps, a continuum of boxes starting out as a flat base with no height and morphing into a folded sheet of paper. Developing a knack for picturing such phenomena takes time and practice. Physical or computational manipulatives can help students develop this knack, but they are no substitute for the ability to create and perform thought experiments on one's own. One goal of our curriculum materials is to make explicit the kinds of thought experiments that bring life to such contexts.

Finding, Articulating, and Explaining Patterns

Finding regularity, especially subtle regularity, is a prized and useful skill in mathematics. So is articulating in precise language what one sees. And, of course, explaining why things happen is at the heart of doing mathematics.

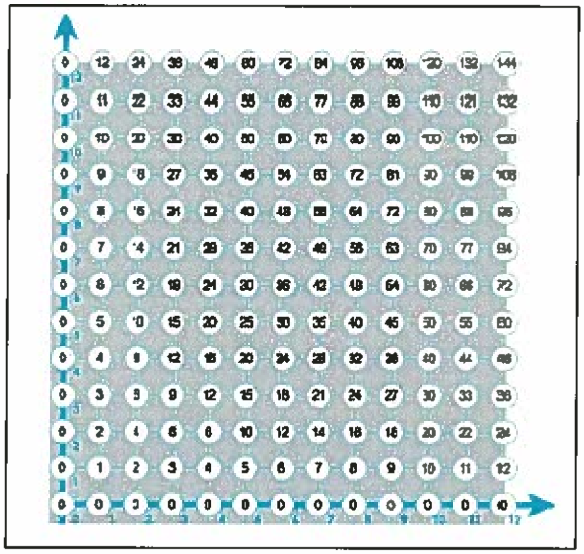

Fig. 1 Finding, describing, and explaining patterns is an important habit of mind.

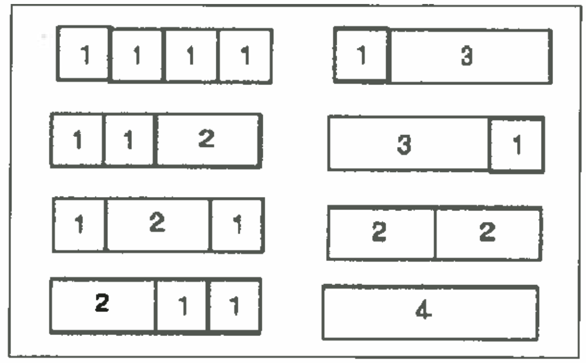

Fig. 2 Eight trains of length 4 can be made with Cuisenaire rods.

For example consider the portion of the multiplication table (see **fig. 1**), oriented in a way that agrees with the typical orientation of Cartesian coordinates (EDC 2009). Readers are invited to—

- find a recurring pattern in the table;
- describe that pattern in precise mathematical language; and
- explain why that pattern exists.

Creating and Using Representations

Much has been written about the problem of counting the number of "trains" of a given length (see **fig. 2**) that can be made from Cuisenaire rods® (EDC 2003; Pagni 1998; Parker 1991). Many variations on the problem exist—using only rods of specified length, counting the number of "cars" of a specified size in the collection of trains of a given length, and many others. We have seen some very clever approaches and solutions to the problem. But all the solutions start with a representation of the context. Different representations suggest different approaches—finding algorithms to generate all the possibilities, lining up all the rods of length 1 to make the specified length and inserting dividers, making all the trains of length n from trains of length $n - 1$ in a systematic way, and a host of others. The mathematical habit of *representing*—mapping a new situation into one that is better understood (NCTM 1991)—is ubiquitous in mathematics and is a focus of our curricula.

Generalizing from Examples

Often students look at a problem and have no idea how to start. A common mantra many teachers use is, "Try it with numbers." Mathematics is open to experiment, and general results or at least conjectures often spring from trying specific examples, looking for regularity, and seeking what seem to be general trends. For example, we ask first-year algebra students to think about which positive integers can be represented as a *difference* of two perfect squares. This question can lead to a lovely result, but there is another reason to include it in an algebra course: It helps students develop the habit of generalizing from examples.

Articulating Generality in Precise Language

Another of our investigations asks students to consider the question, "Which integers can be written as the *sum* of two perfect squares?" Within twenty minutes or so, many students are able to tell whether various integers can or cannot be so written, and many can predict without actually trying all the possibilities. But it takes another half hour for a group to arrive at a consensus about how to describe its tests in precise mathematical language. An often-overlooked fact is that one can understand something without being able to articulate this understanding, and the habit of shoehorning insights into precise mathematical language takes time to develop. One device that we have found very effective in developing this skill is to include recurring dialogues among a small set of characters with different personalities and strengths—students who struggle with how to say what they know. By reading or acting out these dialogues, students begin to have similar conversations with their classmates, and they gradually develop this habit of articulating their ideas in precise language.

Expecting Mathematics to Make Sense

This is a central message of NCTM's *Focus in High School Mathematics: Reasoning and Sense Making* (2009), which provides many examples from the high school curriculum. Another example comes from the algebra companion to that document, *Reasoning and Sense Making in Algebra* (NCTM 2009a): A triangle is determined by its three side lengths. It makes sense, then, that one should be able to find

the area of a triangle from the lengths of its sides. Finding that area is not a simple task, but this insight leads one to believe that a *formula* for the area of a triangle in terms of its side lengths should exist. It does, and algebra can lead one to Heron's formula. Along these lines, a triangle is also determined by the lengths of its three medians. So there should be a formula for the area of a triangle in terms of these lengths. Readers are invited to find one.

Analytic and Geometric Habits of Mind

Certain habits of mind seem to be more prevalent in specific branches of mathematics. Thinking about continuous variation, dynamically changing systems, continuous deformations, and continuous functions are all much more common in geometry, analysis, and physics than they are in, say, algebra and combinatorics. That is not to say that algebra is devoid of considerations about change, but we have found that geometry and "precalculus" are much more suited to the development of this kind of thinking. Consider the following examples.

Reasoning by Continuity

The mathematician Thomas Banchoff asks his advanced calculus students on the first day of class, "Was there a time in your life when your height in inches was equal to your weight in pounds?" One approach is to try to find such a time. Another is to invoke continuity and an informal appeal to the thinking behind (and not necessarily the statement of) the intermediate value theorem of calculus.

Seeking Geometric Invariants

Hand in hand with the habit of reasoning about continuously changing systems is the habit of seeking *invariants* in those systems—things that *do not* change (Cuoco and Goldenberg 1997).

Looking at Extreme Cases and Passing to the Limit

A chapter in our geometry book was inspired by an insight of a high school student many years ago. On a standardized test, Rich was given an equilateral triangle of side length 10 and a point P somewhere in the triangle's interior. The problem was to find the sum of the distances from that point to the sides of the triangle. A theorem about this exists, but Rich did not know it. Instead, he reasoned that, because the question did not say otherwise, he could place P anywhere in the triangle's interior—he assumed that the sum of the distances from P to the sides of the triangle is constant. And then, in his mind, he tucked P into a corner of the triangle, where two of the distances go to zero and the other approaches the height of the triangle. Passing to the limit, he had a number he could calculate. Dynamic geometry allows Rich's thought experiment to become a real one, which is elaborated in Cuoco, Goldenberg, and Mark (1995). Again, the result—that the sum of the distances is constant—and the ensuing corollaries and generalizations are interesting in their own right, but another reason for putting this investigation into a high school program is that it supports the habit of looking to extremes.

Modeling Geometric Phenomena with Continuous Functions

One way to think about the insight shown in the previous example is to consider a function defined on the interior of a triangle whose value at a point is the sum of the distances to the sides of the triangle. The assumption that Rich made informally is that this function is constant on the triangle's interior. Continuity and analytic thinking, based in the structure of the real numbers, have always been just under the surface in plane geometry, going back to Euclid (Cuoco and Goldenberg 1997).

Algebraic Habits of Mind

Just as certain mathematical habits are indigenous to geometry and analysis, algebraists use certain ways of thinking quite often. These have to do with

> **The habit of shoehorning insights into precise mathematical language takes time to develop.**

finding patterns in calculations, expressing regularity as algorithms, and looking for structural similarities among various systems in which one can calculate. Consider the following six examples.

Seeking Regularity in Repeated Calculations

This habit manifests itself when one is performing the same calculation over and over and begins to notice the rhythm in the operations. Seeking and articulating this regularity is a backbone of algebraic thinking (Cuoco, Goldenberg, and Mark 2009). The following brief example is developed more fully in the *CPE Project: Algebra 1* (EDC 2009, p. 473):

> Suppose you want to buy a music CD, and a web site offers a 28% discount on the list price. It also adds a 5% state sales tax and a $3.50 shipping charge. The local music store sells CDs for 10% of list price, also charges 5% sales tax, but has no shipping charge. Ignoring convenience and drive time, for which list prices is it better to go online?

There are many ad hoc ways of solving such problems, but a core algebraic approach is to develop a function or algorithm that will allow one to compute the price of a CD in each scenario easily. For example, when buying from the Web site, one determines the total cost of a CD by taking the list price, subtracting 28% of that amount, adding 5% of that amount, and then adding \$3.50. This may sound simple enough, but many students are not able to articulate this process without referring to a specific price, like \$25. In other words, they can compute the total cost of a CD for any particular list price but cannot express the general process for calculating the total cost of a CD. And expressing that process is precisely the first step of what is needed to find the break-even point. One way to develop the skill of expressing this kind of generality is to begin with some numerical examples until one gets the rhythm of the calculations and is able to articulate the cost of a CD in a way that will work for any price.

Delayed Evaluation—Seeking Form in Calculations

General patterns often get masked when expressions or numerical calculations are evaluated too early. Often in algebra one wants to delay numerical evaluation until the end of a process so that one can see how the operations are sequenced and so that the structure of the calculation (rather than its value) becomes more apparent. For more about delayed evaluation, see NCTM (2009b) and Cuoco (2005).

Chunking—Changing Variables to Hide Complexity

Often in algebra one wants to treat a whole expression as a single object. Teachers often cover up parts of an expression with a hand and ask students to think of what is under the hand as a single entity. For example, $x^4 + 2x^2 + 1$ can be considered as a quadratic in x^2:

$$x^4 + 2x^2 + 1 = (x^2)^2 + 2(x^2) + 1 = \clubsuit^2 + 2\clubsuit + 1$$

Chunking has applications throughout high school algebra. Further examples are described in Cuoco (2008) and NCTM (2009b).

Reasoning about and Picturing Calculations and Operations

A key habit of mind in algebra involves predicting how a calculation will go without having to carry it out. For example, consider the following question: "When is the average of two averages the average of the whole lot?" After a little experimentation, one begins to reason about the *process* of averaging and comes up with several situations in which the statement is true and several more when it is not. More examples of this habit are explicated in NCTM (2009a) and National Governors Association Center for Best Practices and Council of Chief State School Officers (2009).

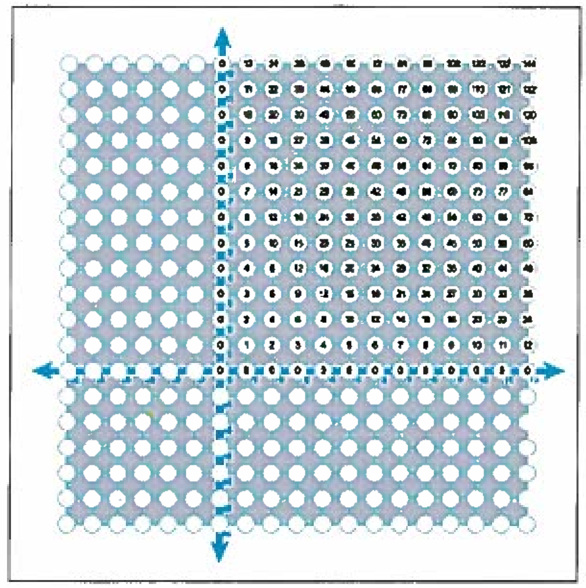

Fig. 3 How can the blanks be filled so as to preserve the patterns and the calculation rules?

Extending Operations to Preserve Rules for Calculating

Suppose we extend the multiplication table shown in **figure 1** to the other quadrants (see **fig. 3**). How could we fill in the blanks to maintain patterns in the rows and columns? There are many answers to this question because there are many ways to describe patterns in the rows and columns. But, in fact, there is only *one* way to complete the table that will preserve the basic rules for calculating with positive integers (e.g., the distributive law).

This habit of extending to preserve rules is a key algebraic habit of mind, one that runs throughout our programs. The point is that extensions of algebraic operations are not arbitrary—they are forced on us by the desire to maintain the basic rules of arithmetic.

Purposefully Transforming and Interpreting Expressions

Focus in High School Mathematics: Reasoning and Sense Making (NCTM 2009a) uses the phrase *mindful manipulation* when it refers to the habit of transforming algebraic expressions to reveal hidden meaning. Writing $3x^2 - 12x + 16$ as $3(x - 2)^2 + 4$ tells one that, when x is replaced by any real number, the value of the expression is never smaller than 4. Writing $15x^2 - 57x + 46$ as

$$5(x - 1)(x - 2) + 2(x - 2)(x - 3) + 8(x - 1)(x - 3)$$

makes it easy to evaluate the quadratic at 1, 2, and 3. Writing Heron's formula in the form

$$A = \left(\frac{1}{4}\right)\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}$$

helps one see exactly when the formula produces zero (NCTM 2009b). Finding examples where purposeful transformations produce added value and utility is an important part of our curriculum development efforts.

Seeking and Specifying Structural Similarities

Watch high school juniors and seniors calculate with complex numbers in the form $a + bi$. Many act as if i is x and complex numbers are just polynomials. Somewhere in the calculation, they replace i^2 with -1. Without using any formal language, these students are noticing that the system of complex numbers behaves like the system of polynomials in one variable with real coefficients, with an extra simplification rule ($i^2 = -1$). Students are noticing a close similarity between the algebraic structures of the complex numbers and polynomials over the real numbers.

Developing the habit of looking for and describing these structural similarities is one of the basic habits in modern algebra, and the high school curriculum provides many opportunities to highlight it. Just a few examples: Factoring polynomials is quite a bit like factoring integers; 2×2 matrices and their multiplication have the same structure as linear transformations of the plane under composition; and roots of unity behave like polynomials with a certain remainder arithmetic.

CONCLUSION

One of the most satisfying aspects of this habits-of-mind focus is that the authors were able to connect many seemingly different topics from the high school curriculum by looking beneath the topics to find approaches that revealed remarkable conceptual similarities. This approach not only brought coherence to the program; it also brought parsimony. By focusing on the underlying habits of mind needed to deal with the topics, we were able to drastically reduce the method bloat that plagues high school mathematics, providing students with a few general-purpose tools that could be applied across the entire curriculum (Cuoco 2008).

Space prohibits their inclusion here, but mathematical habits of mind allow one to bring coherence to many different parts of the curriculum; see Cuoco, Goldenberg, and Mark (2009).

When we began this line of work—organizing curricula around mathematical thinking—our goal was to make school mathematics both more accessible to high school students and more closely aligned with mathematics as a scientific discipline. Once implemented, the method had other benefits—

coherence among topics and parsimony of methods. In hindsight, these benefits and results should have been obvious. The results and methods of mathematics are its artifacts; the actual mathematics lies in the thinking that creates the artifacts.

ACKNOWLEGMENTS

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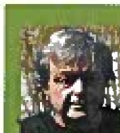

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